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Flow induced by jets and plumes

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The order of magnitude of the flow velocity due to the entrainment into an axisymmetric, laminar or turbulent jet and an axisymmetric laminar plume, respectively, indicates that viscosity and non-slip of the fluid at solid walls are essential effects even for large Reynolds numbers of the jet or plume. An exact similarity solution of the Navier–Stokes equations is determined such that both the non-slip condition at circular–conical walls (including a plane wall) and the entrainment condition at the jet (or plume) axis are satisfied. A uniformly valid solution for large Reynolds numbers, describing the flow in the laminar jet region as well as in the outer region, is also given. Comparisons show that neither potential flow theory (Taylor 1958) nor viscous flow theories that disregard the non-slip condition (Squire 1952; Morgan 1956) provide correct results if the flow is bounded by solid walls.

1. Introduction

It is a common assumption that the flow induced by a slender (i.e. high-Reynolds-number) jet is, in a first approximation, an inviscid potential flow. Taylor (1958) and Kraemer (1971) calculated such flows for various conditions (two-dimensional and axisymmetric, forced and buoyant, with and without walls). On the other hand, exact solutions for laminar, axisymmetric jets emerging from a hole in a plane or conical wall were given by Squire (1952) and Morgan (1956), respectively. Their solutions satisfy the boundary condition of zero velocity normal to the wall but not the condition of zero tangential velocity. Nevertheless Squire (1952) noted that the inflow into the laminar jet is a viscous flow even in the limiting case of a very high speed jet.

Recently Potsch (1980) reconsidered and generalized the exact solutions for axisymmetric laminar jets with arbitrary Reynolds numbers $Re = \bar{M}^{\frac{1}{2}}/\nu$ (kinematic momentum flux $2\pi\bar{M}$ and kinematic viscosity ν). Studying the limiting case $Re \to \infty$, Potsch showed that in the presence of walls the flow induced by the jet does not agree with the results of the potential flow theory. But, as in Squire's (1952) and Morgan's (1956) investigations, there is a non-zero tangential velocity at the wall in Potsch's solutions. Furthermore, both Morgan and Potsch state that a non-trivial similarity solution for the jet flow satisfying the non-slip condition at conical walls (including the plane wall normal to the jet axis) does not exist.

In this paper it is shown that a non-trivial similarity solution satisfying the non-slip condition at the walls exists in the limiting case $Re \to \infty$. The solution is given

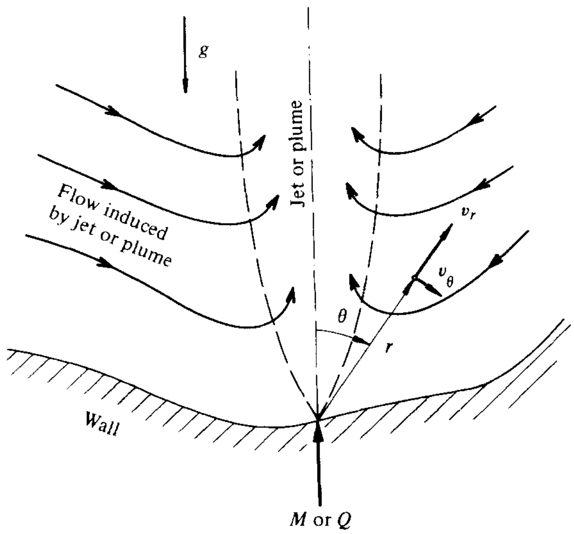

FIGURE 1. Flow regions and co-ordinates.

for the axisymmetric flow induced by a (forced) jet or a plume. The results are remarkably different from those commonly presented in the literature (e.g. Schlichting 1979, figures 11.5 and 24.10 or Rosenhead 1963, figure III.11).

2. The importance of viscosity in the outer flow region

Consider an axisymmetric slender jet emerging from a point source of momentum flux M or an axisymmetric slender plume emerging from a point source of heat flux Q (figure 1). It is convenient to introduce a kinematic momentum flux $\bar{M}$ and a buoyancy flux $\bar{Q}$ according to the relations

$$\bar{M} = M/2\pi\rho, \quad \bar{Q} = g\beta Q/2\pi\rho c_p, \tag{1}$$

with constant values of density ρ, specific heat capacity c_p, thermal expansivity β, and gravity acceleration g. Owing to the coefficients 2π in equation (1) both $\bar{M}$ and $\bar{Q}$ are referred to the unit azimuthal angle. Only steady flows will be considered.

It follows from the conservation equations of mass, momentum and energy that the volume entrainment rate per unit of length of both the laminar jet and the laminar plume is equal to $8\pi K\nu$ where K is a dimensionless constant and ν is the kinematic viscosity. Schlichting's solution of the boundary-layer equations for the laminar jet gives $K = 1$ (cf. Schlichting 1979, p. 233). In case of a laminar plume, analytical solutions for Prandtl numbers $Pr = 1$ and $Pr = 2$ are known, and numerical results for some other values of Pr are also available (Fujii 1963; Mollendorf & Gebhart 1974). The values of K obtained from those results are given in table 1. The turbulent jet, but not the turbulent plume, also has a constant entrainment rate per unit length. According to Schlichting (1979) p. 749, it is given by $0{\cdot}404(2\pi\bar{M})^{\frac{1}{2}}$. If this is, as in the laminar case, put equal to $8\pi K\nu$ we obtain $K = 0{\cdot}0403\,Re$ with jet Reynolds number $Re = \bar{M}^{\frac{1}{2}}/\nu$.

Because of entrainment, a flow is induced in the region outside the jet or plume. As far as the general discussion in this section is concerned there are no restrictions

	Laminar jet	Turbulent jet	Laminar plume $Pr = 0{\cdot}01$	$0{\cdot}7$	1	2	7	10
K	1	$0{\cdot}0403\, Re$	120	$1{\cdot}88$†	$1{\cdot}5$	1	$0{\cdot}77$	$0{\cdot}67$
		$(Re = \overline{M}^{\frac{1}{2}}/\nu)$	—	$1{\cdot}98$‡	—	—	—	—

† According to Fujii (1963).
‡ According to Mollendorf & Gebhart (1974).

TABLE 1. Entrainment constant K.

with respect to the shape of the wall or the symmetry of the induced flow, i.e. it can be any three-dimensional flow. Furthermore, the induced flow can be considered a laminar one even if the jet flow is turbulent since the turbulent jet is bounded by a well-defined (though irregularly moving) surface across which the entrainment of non-turbulent fluid occurs (cf. Rotta, 1972, p. 162; Townsend 1976, p. 209). With respect to the outer flow region, the slender laminar or turbulent jet and the laminar plume, respectively, act as a line mass sink of constant strength. Thus, the induced flow has to satisfy the following boundary conditions at the jet or plume axis in terms of spherical co-ordinates r, θ with velocity components v_r, v_θ (figure 1):

$$\lim_{\theta\to 0} (\theta v_\theta) = -4K\nu/r; \tag{2a}$$

$$\lim_{\theta\to 0} \left(\frac{\partial v_r}{\partial \theta}\right) = 0. \tag{2b}$$

We may mention that from the point of view of the method of matched asymptotic expansions, equations ($2a$, b) are conditions for matching the inner solution (jet or plume flow) with the outer solution (induced flow).

Equation ($2a$) shows that in the outer flow region, where θ is of order 1, v_θ is of order $K\nu/r$. Furthermore, the entrainment into the jet (or plume) for a length r has to be balanced by a radial volume flux through a spherical surface of radius r. Hence v_r is also of the order $K\nu/r$ in the outer flow region. With the characteristic length of order r we finally obtain that the *local* Reynolds number $Re^* = |\mathbf{v}|r/\nu$ is of the order of K in the outer region. Viscosity is expected to be of importance if Re^* is of order 1.

Let us now recall that $K = 1$ for a laminar jet and $K = O(1)$ for a laminar plume except in the limiting case $Pr \to 0$; cf. table 1. Therefore, viscosity is important in the whole flow field induced by an axisymmetric laminar jet or plume no matter how large the Reynolds number of the jet ($Re = \overline{M}^{\frac{1}{2}}/\nu$) or plume ($Re = \overline{Q}^{\frac{1}{4}}r^{\frac{1}{2}}\nu^{-\frac{3}{4}}$) may be. For a turbulent jet, however, the jet Reynolds number must not be too large (say < 500) in order to ensure that K remains of order 1, cf. table 1. In this case, too, the (time-averaged) outer flow is a viscous one.

The differential equations governing the viscous outer flow are the full Navier–Stokes equations. If there are no solid walls bounding the flow, solutions of the potential flow equation (Laplace's equation) can satisfy the Navier–Stokes equations and all boundary conditions of the outer flow. Thus the solutions given by Taylor (1958) or Kraemer (1971) are the correct ones in case of an unbounded fluid. If, however, the fluid is bounded by solid walls (figure 1), solutions of the potential flow equation cannot, in general, satisfy the non-slip condition at the walls. In this case the full Navier–Stokes equations subject to appropriate boundary conditions, i.e. non-slip conditions at the walls and equations ($2a$, b) at the jet axis, have to be solved in order to find the outer flow field.

It should be emphasized that the conclusions drawn so far are valid for axisymmetric free jets (or plumes) only. In the case of plane flow as well as in the case of an axisymmetric wall jet it turns out that the local Reynolds number in the outer flow is of the same order of magnitude as the Reynolds number in the jet flow. If the latter is very large, boundary-layer equations apply to the jet flow, whereas viscosity is negligible in a first approximation of the outer flow.

3. The wall shear stress as a second-order term in the momentum balance

When an attempt is made to solve the Navier–Stokes equations for the flow induced by an axisymmetric jet (or plume) one might recall Morgan's (1956) and Potsch's (1980) statement on the non-existence of an appropriate similarity solution for arbitrary (but finite) jet Reynolds numbers. Potsch (1980) also indicated that the solution cannot exist because, due to a singularity at the jet origin, the viscous force at the wall would be infinite, which contradicts the finite momentum source. In the limit of an infinite jet Reynolds number, however, the induced (outer) flow becomes singular at the jet axis. This resembles the swirling vortex flow studied by Serrin (1971), where a similarity solution satisfying the non-slip condition at the wall exists if, and only if, the flow is singular at the axis.

Therefore an investigation of the shear stress at the wall seems to be in order. For what follows it will be assumed that $K = O(1)$. Since the characteristic length in the outer region is of order r and the velocity is of order ν/r, the order of magnitude of the wall shear stress τ_w is given by

$$\tau_w/\rho = (\nu/r)^2 O(1). \tag{3}$$

Thus the viscous force F exerted on a wall of surface area A becomes

$$F/\rho = \nu^2 \iint_A \frac{dA}{r^2} O(1). \tag{4}$$

With dA proportional to $r\,dr$, the integral of (4) has a non-integrable singularity at $r = 0$ which leads to an infinite force. Furthermore, if the wall extends to infinity, the integral diverges also as $r \to \infty$. But because the force is proportional to ν^2, it is of the order of Re^{-2} when compared with the kinematic momentum flux $\bar{M}$ from the source. As $Re \to \infty$, the viscous wall force is a higher-order term which need not be taken into account in a first-order momentum balance. This only justifies the usual assumption that the momentum flux is constant in a slender jet even if the outer flow field is bounded by solid walls. A higher-order theory will have to deal with the region near $r = 0$, but this is beyond the scope of the present paper. It should be mentioned, however, that a similar problem arises in the boundary-layer theory for a flat plate, where the second-order wall shear stress has a non-integrable singularity at the leading edge (cf. Van Dyke 1975, p. 137).

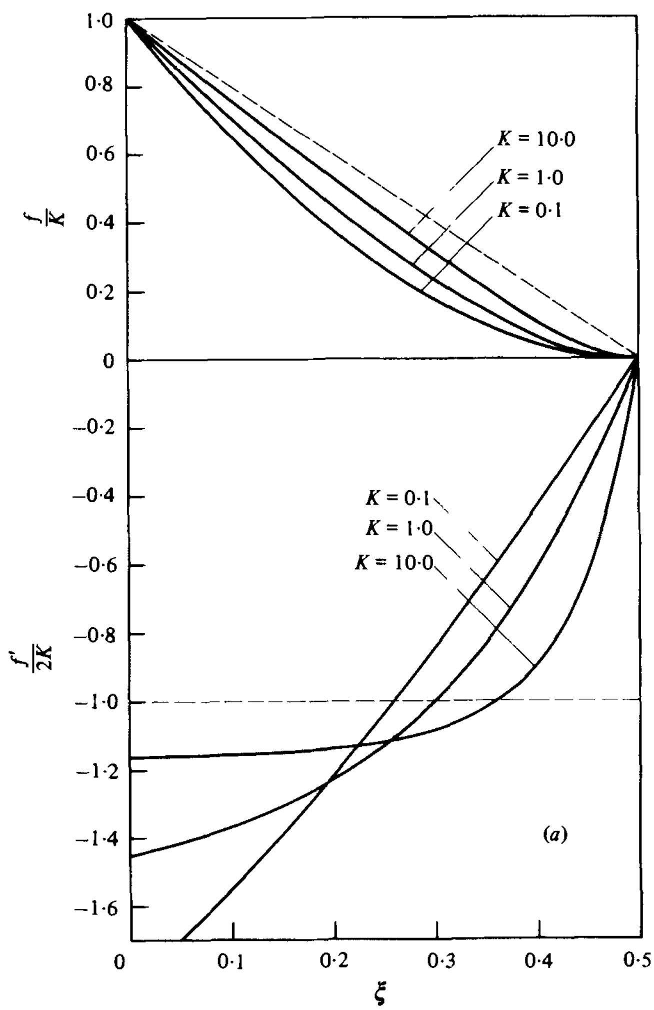

FIGURE 2(*a*). For legend see next page.

4. Solution for axisymmetric outer flows bounded by conical walls

Consider an axisymmetric jet or plume emerging from the apex of a circular-conical wall with axes ($\theta = 0$) of cone and jet or plume coinciding. Let the semi-vertex angle of the cone be θ_w, with $0 < \theta_w < \pi$. If $\theta_w = \frac{1}{2}\pi$, the jet emerges from a plane wall. The limiting case $\theta_w \to \pi$ corresponds to a jet emerging from the apex of a very slender, conical tube.

It will be convenient to use spherical co-ordinates with a new variable ξ defined by

$$\xi = \tfrac{1}{2}(1-\cos\theta). \tag{5}$$

At the wall we have $\xi = \xi_w$, with

$$\xi_w = \tfrac{1}{2}(1-\cos\theta_w). \tag{6}$$

Introducing a Stokes' stream function ψ in order to satisfy the continuity equation, eliminating the pressure from the Navier–Stokes equations, separating the variables by writing

$$\psi = 4\nu r f(\xi) \tag{7}$$

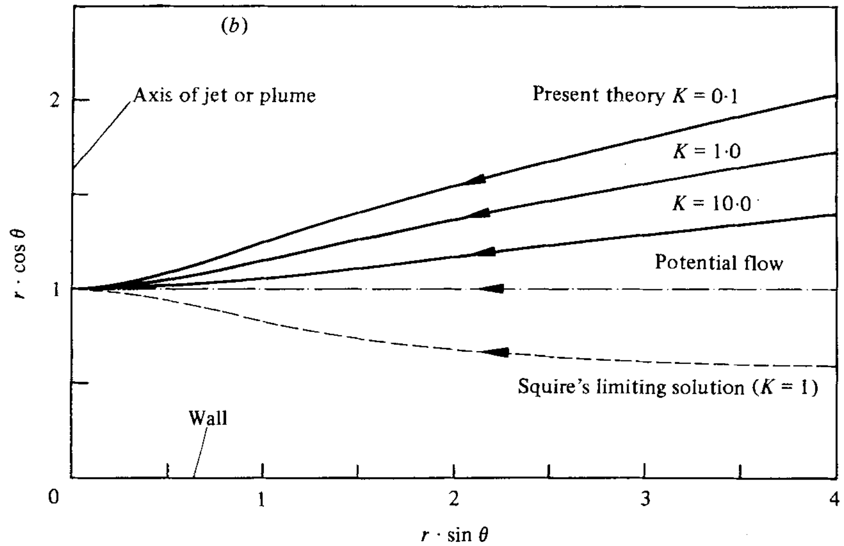

FIGURE 2. Flow induced by an axisymmetric, slender jet or plume emerging from a plane wall perpendicular to the axis of the jet or plume ($\theta_w = 90°$). Regarding K cf. table 1. (a) Dimensionless stream function and its derivative: ——, present theory; – – –, potential flow theory. (b) Representative streamlines (arbitrary unit of length).

and integrating three times we obtain, according to Batchelor (1970), p. 207, the following ordinary differential equation for the dimensionless stream function $f(\xi)$:

$$\xi(1-\xi)f' - (1-2\xi)f + f^2 = C_0 + C_1\xi + C_2\xi^2; \tag{8}$$

C_1, C_2 and C_3 are arbitrary constants of integration, and primes denote derivatives with respect to ξ. Since the velocity components in terms of the dimensionless stream function are

$$v_r = \frac{2\nu}{r} f'(\xi), \quad v_\theta = -\frac{2\nu}{r}\frac{f(\xi)}{[\xi(1-\xi)]^{\frac{1}{2}}}, \tag{9}$$

the boundary conditions at the wall become

$$f(\xi_w) = 0, \quad f'(\xi_w) = 0. \tag{10}$$

Further boundary conditions are provided by (2a) and (2b) which yield

$$f(0) = K \tag{11}$$

and

$$\lim_{\xi \to 0} [\xi^{\frac{1}{2}} f''(\xi)] = 0. \tag{12}$$

The latter condition is satisfied if $f''(\xi)$ remains bounded as $\xi \to 0$. From the boundary conditions (11) and (10), we obtain, respectively,

$$C_0 = K(K-1) \tag{13}$$

and

$$C_2 = -(C_0 + C_1\xi_w)/\xi_w^2. \tag{14}$$

Differentiating equation (8) with respect to ξ, the condition (12) may be used to derive the relation

$$C_1 = 2K[1 + f'(0)]. \tag{15}$$

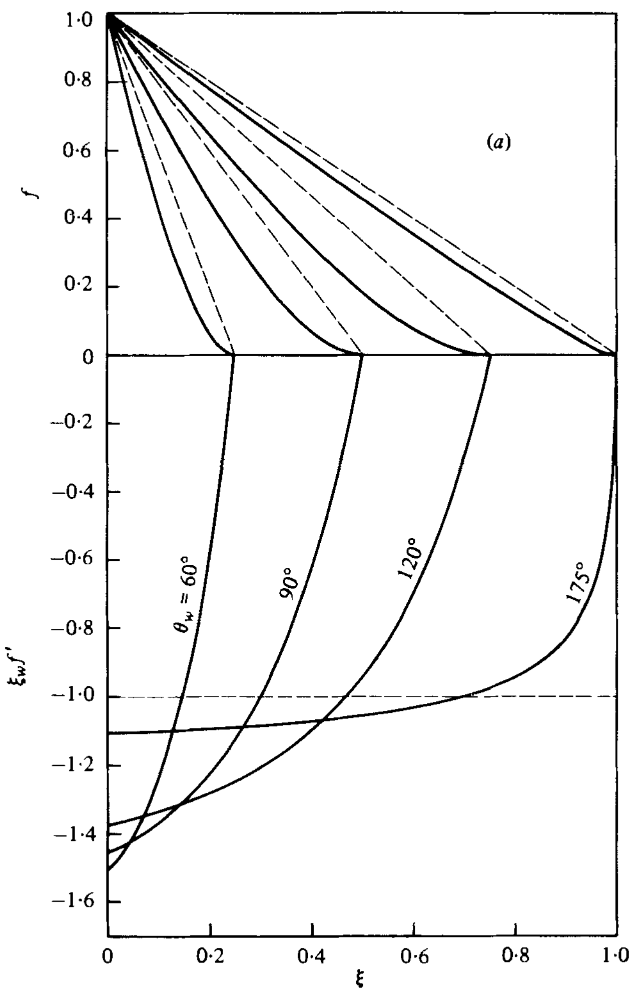

FIGURE 3(*a*). For legend see next page.

The problem can now be stated as follows: Determine $f'(0)$ such that the solution of the first-order differential equation (8) with constants C_0, C_1 and C_2 given by equations (13) to (15) satisfies the boundary condition $f(\xi_w) = 0$. This problem has been solved for various values of K and ξ_w by a Runge–Kutta shooting method. Some results are presented in figures 2(*a*) and 3(*a*).

For the purpose of comparisons it should be mentioned that in an inviscid (potential) flow theory the linear terms of equation (8) are omitted. Retaining only the nonlinear term and disregarding the second one of the boundary conditions (10), we obtain the solution

$$f = K(1 - \xi/\xi_w) \quad \text{(potential flow).} \tag{16}$$

This is in agreement with Taylor's (1958) results (for $\xi_w = \frac{1}{2}$ and $\xi_w = 1$).

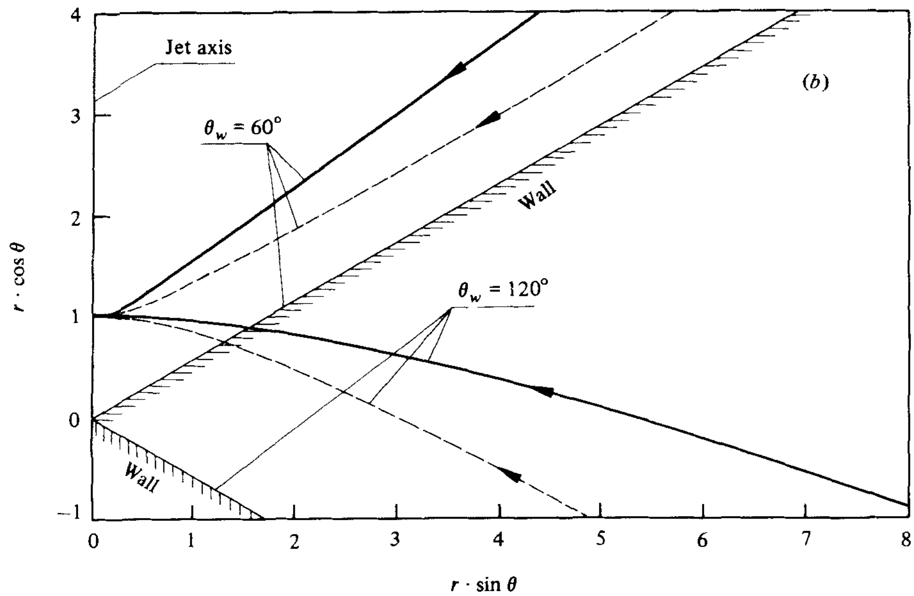

FIGURE 3. Flow induced by an axisymmetric, laminar slender jet or plume (with $Pr = 2$ in case of the plume) emerging from the apex of right conical walls with various semi-vertex angles θ_w. (a) Dimensionless stream function and its derivative. (b) Representative streamlines (arbitrary unit of length). ——, present theory; – – –, potential-flow theory.

5. Results

Using the results of the numerical integration, which are summarized in figures 2(a) and 3(a), the velocity components can easily be determined from (9). Besides, the flow field can be illustrated by drawing the streamlines ψ = const. which are obtained from (7). Since, according to (7), r is proportional to ψ if ξ (or θ) is kept constant, it is sufficient to draw only one representative streamline. Some results are presented in figures 2(b) and 3(b).

The diagrams clearly show that potential flow theory does not provide correct results for outer flows bounded by solid walls. Note that the streamlines of the potential flows are always too close to the walls when compared to the correct solutions. This can be explained by the decrease of the radial flow velocity near the walls due to viscosity effects.

It is remarkable that Squire's (1952) solution, which is an exact solution of the Navier–Stokes equations but does not satisfy the non-slip condition at the wall, is even more unrealistic than the potential flow theory, cf. figure 2(b).

It can be seen from figure 3(a), however, that in the limiting case of a very slender tube ($\xi_w \to 1$, $\theta_w \to \pi$) the influence of the wall is felt only in a thin, axisymmetric layer, where $f''(\xi)$ becomes very large. Besides, viscosity effects are, of course, also confined to a thin boundary layer at the wall if the entrainment constant K is very large since this results in very large values of the local Reynolds number Re^*; cf. figure 2(a), $K = 10$.

From the point of view of the method of matched asymptotic expansions, the results presented so far are outer solutions of laminar flows as $Re \to \infty$. Inner solutions are

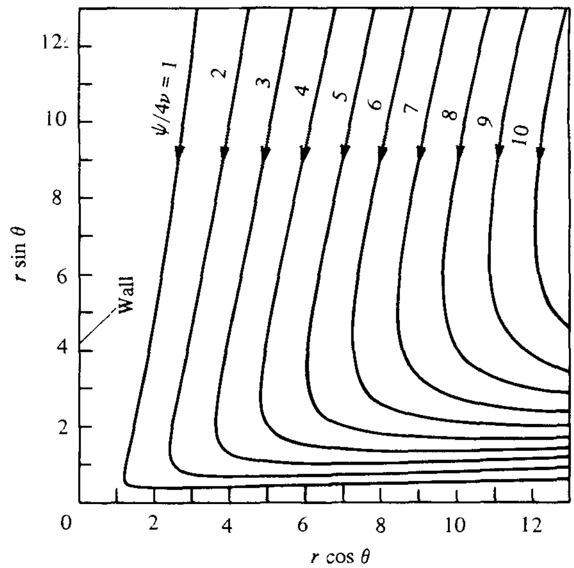

FIGURE 4. Streamlines of an axisymmetric laminar jet flow emerging from a point source in a plane wall perpendicular to the jet axis ($\theta_w = 90°$). Uniformly valid first approximation, $Re = 20$, arbitrary unit of length.

provided by the well-known similarity solutions of the boundary-layer equations for laminar jets and plumes. In case of the jet there is an analytic solution (cf. Schlichting 1979, p. 230), which is of the same similarity structure as our outer solution according to (7). In terms of the stretched (inner) co-ordinate $\tilde{\xi}$ defined by

$$\tilde{\xi} = \tfrac{3}{8} Re^2 \xi, \quad Re^2 = \overline{M}/\nu^2 \tag{17}$$

the inner solution is

$$f_i = \tilde{\xi}/(1+\tilde{\xi}). \tag{18}$$

We can now find a uniformly valid solution f_+ by adding the inner solution f_i to the outer solution f, and subtracting the common part of the two expansions, i.e. $f(0)$. Since $f(0) = K = 1$ for a laminar jet, we obtain

$$f_+ = f - (1 + \tfrac{3}{8} Re^2 \xi)^{-1} \quad \text{(laminar jet, } Re \gg 1). \tag{19}$$

Streamlines $\psi = 4\nu r f_+(\xi) = \text{const.}$ are shown in figure 4 for a typical example. Note that the distance between neighbouring streamlines near the wall is distinctly larger than between corresponding streamlines further away from the wall. This is, of course, a consequence of the fact that the velocity vanishes as the wall is approached. Thus the streamline pattern shown in figure 4 is quite different from the well-known picture obtained by Squire (1952) with a solution that does not satisfy the non-slip condition at the wall. We refrained from including Squire's result in figure 4 since in the outer region the uniformly valid solution is equivalent to the outer flow field solution which was already compared with the results of other theories in figure 2(*b*).

6. Conclusions

(1) Although slender jets or plumes with very large Reynolds numbers are considered, i.e. $Re \gg 1$, convective terms and viscous terms are of the same order of magnitude in the outer flow that is induced by

(*a*) an axisymmetric laminar jet;

(*b*) an axisymmetric laminar plume with $Pr^{-1} = O(1)$;

(*c*) a turbulent jet with an entrainment constant $K = O(1)$.

(2) If there are no walls, solutions of the potential equation can satisfy the viscous-flow equations (Navier–Stokes equations) and all boundary conditions in the outer flow region (Taylor 1958; Kraemer 1971).

(3) If the viscous flow is bounded by solid walls the non-slip condition has to be satisfied at the walls. Since this cannot in general be accomplished by solving the potential equation, the full Navier–Stokes equations have to be solved in the outer flow region.

(4) Nevertheless an essential simplification is gained by considering an asymptotic expansion in terms of large Re. For, in case of $Re = O(1)$ no similarity solution that gives a finite momentum flux from the jet source *and* satisfies the non-slip condition at the wall can be found (Morgan 1956; Potsch 1980).

(5) Similarity solutions for outer flows bounded by conical walls (including a plane wall) give velocity distributions and streamline patterns that differ appreciably from solutions obtained previously. Neither potential flow theory (Taylor 1958) nor viscous-flow theories that disregard the non-slip condition (Squire 1952; Morgan 1956) provide correct results if the flow is bounded by solid walls.

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